Lecture 6: Exact Intersections

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We deduced from the Frankl-Wilson theorem that if $A \subset [n]^{(r)}$ is such that $|x \cap y| \in L$ for any two distinct $x, y \in A$, where $L$ is a set of $s$ non-negative integers, then

$$|A| \leq \binom{n}{s}.$$ 

What happens for non-uniform families, i.e. subsets of $\mathcal{P}([n])$?

**Definition.** If $L$ is a finite set of integers, a family $A \subset \mathcal{P}([n])$ is said to be $L$-intersecting if $|A \cap B| \in L$ for any two distinct $A, B \in A$.

We ask the following

**Question.** What is the maximum possible size of an $L$-intersecting family of subsets of $\{1, 2, \ldots, n\}$?

We write

$$M(n, L) = \max\{|A| : A \subset \mathcal{P}([n]), A \text{ is } L\text{-intersecting}\}.$$ 

We will see that an $L$-intersecting family of subsets of $\{1, 2, \ldots, n\}$ has size at most

$$\sum_{i=0}^{s} \binom{n}{i},$$

where $s = |L|$. So for $L$ fixed and $n \to \infty$, $M(n, L)$ grows no faster than a polynomial of degree $|L|$. It is a longstanding open problem to determine, in terms of $L$, the rate of growth of $M(n, L)$ as $n \to \infty$ (for each fixed $L$).

**Remark.** Similarly, we define

$$M(n, r, L) = \max\{|A| : A \subset [n]^{(r)}, A \text{ is } L\text{-intersecting}\}.$$ 

Again, it would be interesting to determine, in terms of $L$ and $r$, the rate of growth of $M(n, r, L)$ as $n \to \infty$ (for each fixed $L$ and $r$). It is not even known exactly which $L$ and $r$ force $M(n, r, L)$ to be linear in $n$. For a survey of partial results, see the book by Babai and Frankl [1].

We start off with the simplest case, $|L| = 1$:

**Theorem 1** (Fisher). If $l \in \mathbb{N}$, and $A \subset \mathcal{P}([n])$ is $\{l\}$-intersecting, i.e. $|A \cap B| = l$ for any two distinct $A, B \in A$, then

$$|A| \leq n.$$
Proof. Let $A \subset \mathcal{P}([n])$ be \{l\}-intersecting. First, observe that if there exists $A_0 \in A$ with $|A_0| = l$, then we are done. Indeed, every other set in $A$ then contains $A_0$, so the sets $B \setminus A_0$ (for $B \in A \setminus \{A_0\}$) must be pairwise disjoint, so $|A| \leq n$, as required.

Assume from now on that $|A| > l$ for all $A \in A$. We claim that the characteristic vectors

$$\{\chi_A : A \in A\}$$

form a linearly independent set in $\mathbb{R}^n$. (In fact, this is true in the first case as well, but it is easier to use the above argument.)

To see this, suppose that

$$\sum_{A \in A} c_A \chi_A = 0$$

for some real numbers $\{c_A : A \in A\}$. Taking the (standard Euclidean) inner product with $\chi_B$ gives

$$c_B |B| + \sum_{A \in A \setminus \{B\}} c_A |A \cap B| = 0,$$

and therefore

$$c_B (|B| - l) = -l \sum_{A \in A} c_A \quad \forall B \in A.$$

If $\sum_{A \in A} c_A \neq 0$, then for each $B \in A$, $c_B$ is non-zero and has the opposite sign to $\sum_{A \in A} c_A$, a contradiction. Hence, $\sum_{A \in A} c_A = 0$, so $c_B = 0$ for all $B \in A$, proving the claim.

Since $\mathbb{R}^n$ has dimension $n$, it follows that $|A| \leq n$, proving the theorem. \(\square\)

Remark. We have equality if $l = 1$ and $A = \{\{1\}\} \cup \{\{1, i\} : 2 \leq i \leq n\}$, or if $l = n - 2$ and $A = \left[[n]^{(n-1)}\right]$. We now turn to the case of general $L$.

Theorem 2 (Frankl, Wilson, 1981). Let $A \subset \mathcal{P}[n]$ be $L$-intersecting, where $|L| = s$. Then

$$|A| \leq \sum_{i=0}^{s} \binom{n}{i}.$$

Remark. This is sharp when $L = \{0, 1, \ldots, s - 1\}$: take $A = \left[n\right]^{(s)}$. The proof we give is due to Babai (1988).

Proof. Write

$$A = \{A_1, \ldots, A_m\},$$

where $|A_1| \geq |A_2| \geq \ldots \geq |A_m|$. We will find $m$ linearly independent objects in a vector-space of dimension $\sum_{i=0}^{s} \binom{n}{i}$. Our objects will be real-valued functions on $\{0, 1\}^n$. For each set $A_j$, we will choose a function $f_j : \{0, 1\}^n \rightarrow \mathbb{R}$ such that

$$f_j(\chi_{A_j}) \neq 0 \quad \forall j \in [m],$$

but

$$f_j(\chi_{A_k}) = 0 \quad \forall k > j.$$

This ‘upper-triangular’ condition immediately implies that the $f_j$’s are linearly independent.

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Claim. Let $U$ be a set, and let $f_1, \ldots, f_m$ be real-valued functions on $U$. If there exist points $u_1, \ldots, u_m$ such that
\[ f_j(u_j) \neq 0 \quad \forall j \in [m], \]
but
\[ f_j(u_k) = 0 \quad \forall k > j, \]
then the functions $f_1, \ldots, f_m$ are linearly independent.

Proof of Claim: The $m \times m$ matrix $(f_j(u_k))_{1 \leq j, k \leq m}$ is strictly upper-triangular (upper triangular with all its diagonal entries non-zero). Such a matrix clearly has non-zero determinant, so its rows are linearly independent. In other words, the restrictions $f_j|_{\{u_1, \ldots, u_m\}}$ are linearly independent (as real-valued functions), so certainly the $f_j$’s are. \qed

For each $j$, define $f_j : \{0,1\}^n \rightarrow \mathbb{R}$ by
\[ f(z_1, \ldots, z_n) = \prod_{i \in L: i \in |A_j|} \left( \sum_{i \in A_j} z_i - l \right). \]
We then have
\[ f_j(\chi_{A_k}) = \prod_{i \in L: i \in |A_j|} (|A_j \cap A_k| - l) \quad \forall j, k \in [m]. \]
Hence, $f_j(\chi_{A_j}) \neq 0$ for each $j \in [m]$, but $f_k(\chi_{A_j}) = 0$ whenever $k > j$, as desired. It follows from the claim above that the $f_j$’s are linearly independent as real-valued functions on $\{0,1\}^n$.

Observe that each function $f_j$ can be considered as a polynomial in $\mathbb{R}[X_1, \ldots, X_n]$ with total degree at most $s$. We now perform a process known as multilinearization: the idea is to find a multilinear polynomial $\tilde{f}_j$ with total degree at most $s$, such that $\tilde{f}_j$ agrees with $f_j$ on all of $\{0,1\}^n$. (A multivariate polynomial is multilinear if it has degree at most 1 in each variable.)

For each $j$, express $f_j$ as a real linear combination of monomials of the form
\[ \prod_{i \in T} X_i^{a_i}, \]
where $a_i \geq 1$ for each $i$, and $|T| \leq s$. For each of these monomials, replace it by
\[ \prod_{i \in T} X_i. \]

The resulting polynomial $\tilde{f}_j$ clearly agrees with $f_j$ on all of $\{0,1\}^n$, and is a linear combination of monomials which are products of at most $s$ distinct $X_i$’s, i.e. it is a multilinear polynomial of total degree at most $s$. The vector-space of multilinear polynomials of total degree at most $s$ has dimension
\[ \sum_{i=0}^{s} \binom{n}{i}. \]
indeed, the monomials of the form

$$\prod_{i \in T} X_i \quad (T \in [n]^{(\leq s)})$$

are a basis. Hence, the $f_j$'s live in a vector space of dimension at most

$$\sum_{i=0}^{s} \binom{n}{i},$$

proving the theorem. \qed

References