Lecture 7: Saturated and weakly saturated hypergraphs

David Ellis

1 Saturated hypergraphs

Recall the following

**Definition.** A family \( A \subset \mathcal{P}([n]) \) is said to be an antichain if we never have \( A \subset B \) for any two distinct \( A, B \in A \).

The well-known LYM inequality states that if \( A \subset \mathcal{P}([n]) \) is an antichain, then

\[
\sum_{i=0}^{n} \frac{|A \cap [n]^i|}{\binom{n}{i}} \leq 1.
\]

Lubell proved this by observing that the left-hand side is a probability: it is simply the probability that a maximal chain, chosen uniformly at random, intersects \( A \).

The LYM inequality implies that an antichain in \( \mathcal{P}([n]) \) has size at most \( \frac{n}{\lfloor n/2 \rfloor} \), the size of the ‘middle layer’ in \( \mathcal{P}([n]) \). (This can also be proved by partitioning \( \mathcal{P}([n]) \) into \( \binom{n}{\lfloor n/2 \rfloor} \) disjoint chains.)

Bollobás’ Inequality is a useful extension of the LYM inequality.

**Theorem 1** (Bollobás’ Inequality). Let \( A_1, A_2, \ldots, A_m, B_1, B_2, \ldots, B_m \subset [n] \) such that

\[
A_i \cap B_i = \emptyset \quad \forall i \in [m],
\]

but

\[
A_i \cap B_j \neq \emptyset \quad \forall i \neq j.
\]

Then

\[
\sum_{i=1}^{m} \frac{1}{\binom{|A_i| + |B_i|}{|A_i|}} \leq 1.
\]

**Remark.** Note that the above inequality does not involve \( n \), the size of the ground-set.

**Remark.** Taking \( B_i = A_i^c \) for each \( i \), we recover the LYM inequality.

Lubell generalized his method to prove Bollobás’ Inequality, by showing that the left-hand side is the probability that in a permutation (ordering) of \( \{1, 2, \ldots, n\} \) chosen uniformly at random, all of \( A_i \) comes before all of \( B_i \), for some \( i \).

---

\(^1\)Recall that a maximal chain is a chain of size \( n + 1 \) in \( \mathcal{P}([n]) \), i.e. a family of the form \( \emptyset, \{x_1\}, \{x_1, x_2\}, \{x_1, x_2, x_3\}, \ldots, [n] \)
Definition. We say that a permutation $\sigma \in S_n$ is compatible with a pair of sets $(A_i, B_i)$ if all the elements of $A_i$ come before all the elements of $B_i$ in the ordering $\sigma(1), \sigma(2), \ldots, \sigma(n)$.

Proof of Bollobás’ Inequality. Observe that if $A_1, A_2, \ldots, A_m, B_1, B_2, \ldots, B_m \subseteq [n]$ satisfy the hypotheses of Bollobás’ Inequality, then a permutation $\sigma \in S_n$ cannot be compatible with two distinct pairs $(A_i, B_i)$ and $(A_j, B_j)$ (Indeed, suppose the permutation $\sigma \in S_n$ were compatible with both $(A_i, B_i)$ and $(A_j, B_j)$). Write $\sigma$ as an ordering, $\sigma(1), \sigma(2), \ldots, \sigma(n)$. Without loss of generality, we may assume that $\max\{k : \sigma(k) \in A_i\} < \max\{k : \sigma(k) \in A_j\}$. Since all the elements of $B_j$ come after all the elements of $A_j$ in $\sigma$, all the elements of $B_j$ come after all the elements of $A_i$ in $\sigma$, so $A_i \cap B_j = \emptyset$, a contradiction.)

Moreover, the probability that a uniform random permutation $\sigma$ is compatible with a fixed pair $(A_i, B_i)$ is precisely

$$\frac{1}{\binom{|A_i| + |B_i|}{|A_i|}},$$

since any $|A_i|$-subset of $A_i \cup B_i$ is equally likely to form the first $|A_i|$ elements of $A_i \cup B_i$ in $\sigma$. Hence, the probability that a uniform random permutation in $S_n$ is compatible with one of $\{(A_i, B_i) : i \in [m]\}$ is precisely

$$\sum_{i=1}^{m} \frac{1}{\binom{|A_i| + |B_i|}{|A_i|}};$$

this is at most 1, proving Bollobás’ Inequality.

Bollobás’ Inequality has several applications in extremal combinatorics. It can be used, for example, to determine the minimum number of edges of a $t$-saturated $r$-graph on $n$ vertices. Recall the following

Definition. An $r$-uniform hypergraph (or $r$-graph) is a pair $(V, E)$, where $V$ is a set, and $E \subseteq V^{(r)}$ is a collection of $r$-element subsets of $V$. We call $V$ the ‘vertex-set’ and $E$ the ‘edge-set’.

We denote by $K_t^{(r)}$ the complete $r$-uniform hypergraph on $t$ vertices.

Definition. An $r$-graph is said to be $t$-saturated if it is maximal $K_t^{(r)}$-free. In other words, it contains no $K_t^{(r)}$, but the addition of any $r$-set produces a $K_t^{(r)}$.

For example, a $(2$-) graph is $t$-saturated if it is maximal $K_t$-free. What is the minimum possible number of edges in a $t$-saturated graph on $n$ vertices? When $t = 3$, we are looking at maximal triangle-free graphs. Any complete bipartite graph is maximal triangle-free; in particular, the star on $[n]$ with edge-set $\{\{1, i\} : 2 \leq i \leq n\}$ is a maximal triangle-free graph with $n-1$ edges. It is easy to see that any maximal triangle-free graph on $n$ vertices has at least $n-1$ edges. In general, we have the following

Theorem 2 (Bollobás’ Saturated Hypergraph Theorem). Let $G = (V, E)$ be a $t$-saturated $r$-graph on $n$ vertices. Then

$$|E| \geq \binom{n}{r} - \binom{n - t + r}{r}.$$
**Theorem 3.** Let 
edges. It turns out that again, this is the minimum possible number of edges:

\[ n - \binom{n}{r} + \binom{n-t+r}{r} \]

Moreover, an \((n-r)\)-saturated graph on \(n\) vertices has at least one more copy of \(K_3^r\), so

\[ \left| A_i \right| = \frac{1}{r} \binom{n-t+r}{r} \leq 1, \]

and therefore

\[ |A| \leq \binom{n-t+r}{r}, \]

so

\[ |E| \geq \binom{n}{r} - \binom{n-t+r}{r}, \]

proving the theorem.

\[ \square \]

**2 Weakly saturated hypergraphs**

**Definition.** We say that an \(r\)-graph \(G = (V, E)\) is weakly \(t\)-saturated if there is an ordering of \(V \setminus E\) (the non-edges), say \(A_1, A_2, \ldots, A_m\), such that if we add the non-edges to \(E\) one by one, in this order, at each step we produce at least one more copy of \(K_3^r\).

For example, a \((2-)\)graph is weakly 3-saturated if the non-edges can be added, one by one, in such a way that each addition creates at least one new triangle. It is easy to see that a weakly 3-saturated graph on \(n\) vertices has at least \(n-1\) edges. Moreover, an \((n-1)\)-edge graph is weakly 3-saturated if and only if it is a tree.

In general, every \(t\)-saturated \(r\)-graph is weakly \(t\)-saturated: we can add the non-edges in any order. In particular, the \(r\)-graph \([n]^{(r)} \setminus Y^{(r)}\), where \(|Y| = n - t + r\), is a weakly \(t\)-saturated \(r\)-graph on \(n\) vertices, with \(\binom{n}{r} - \binom{n-t+r}{r}\) edges. It turns out that again, this is the minimum possible number of edges:

**Theorem 3.** Let \(G = (V, E)\) be a \(t\)-saturated \(r\)-graph on \(n\) vertices. Then

\[ |E| \geq \binom{n}{r} - \binom{n-t+r}{r}. \]

To prove this, let \(C_j\) be the vertex-set of a \(K_3^r\) formed when \(A_j\) is added. Then we have \(A_j \subseteq C_j\) for each \(j\), but \(A_k \not\subseteq C_j\) for all \(k > j\). Let \(B_j = C_j^c\), so that

\[ A_j \cap B_j = \emptyset \forall j \in [m], \quad A_k \cap B_j \neq \emptyset \forall k > j. \]

Bollobás’ Inequality does not always hold under this weaker condition (exercise!), but since all the \(A_j\)'s have size \(r\), and all the \(B_j\)'s have size \(n - t\), it suffices therefore to prove the following.
Theorem 4 (Lovász / Frankl / Kalai / Alon). Let $A_1, A_2, \ldots, A_m \subset [n]^{(r)}$ and $B_1, B_2, \ldots, B_m \subset [n]^{(s)}$ such that $A_j \cap B_j = \emptyset$ for all $j \in [m]$, and $A_k \cap B_j \neq \emptyset$ for all $k > j$. Then

$$m \leq \binom{r + s}{r}.$$  

We will prove this in two ways. The first uses only elementary linear algebra, but with some ingenuity. The second uses the machinery of the exterior algebra, but once this has been absorbed, the proof almost ‘writes itself’.

Proof 1: Our aim is to exhibit a linearly independent set of $m$ objects in a vector space of dimension $(r+s)$. To each set $B_j$, we will associate a function $f_j : \mathbb{R}^{r+1} \rightarrow \mathbb{R}$, and to each set $A_j$, we will associate a point $u_j \in \mathbb{R}^{r+1}$; the functions $f_j$ and the points $u_j$ will satisfy the ‘strictly upper-triangular’ condition we had before,

$$f_j(u_j) \neq 0 \forall j \in [m]; \quad f_j(u_k) = 0 \forall k > j.$$  

It will follow immediately that the $f_j$’s are linearly independent as real-valued functions. We will then show that they live in a vector-space of dimension $(r+s)$, completing the proof.

The construction of our functions and our points is slightly more involved than before. Take a set $Y$ of $n$ points in general position in $\mathbb{R}^{r+1}$, meaning that every $(r+1)$-subset of $Y$ is linearly independent. (We can do this, for example, by taking $Y$ to consist of $n$ distinct points on the moment curve \{(1, a, a^2, \ldots, a^r) : a \in \mathbb{R}\} in \mathbb{R}^{r+1}$. The fact that the Vandermonde determinant is non-zero implies that these points are in general position.)

Identify the ground set $[n]$ with the points of $Y$; from now on, we think of the $A_i$’s and the $B_i$’s as subsets of $Y \subset \mathbb{R}^{r+1}$.

Since any $r$-subset of $Y$ is linearly independent, for each $k \in [m]$, we have $\dim(\text{Span}(A_k)) = r$, and therefore $\dim(\text{Span}(A_k)^+) = 1$. Choose any non-zero vector $w_k \in \text{Span}(A_k)^+$. Since the points of $Y$ are in general position, for any $y \in Y$, we have

$$y \in A_k \iff y \in \text{Span}(A_k) \iff \langle y, w_k \rangle = 0.$$  

Hence, $A_k \cap B_j \neq \emptyset$ if and only if there exists some $b \in B_j$ with $\langle b, w_k \rangle = 0$. In view of this, for each $j \in [m]$, define $f_j : \mathbb{R}^{r+1} \rightarrow \mathbb{R}$ by

$$f_j(z_1, \ldots, z_{r+1}) = \prod_{b \in B_j} \langle b, z \rangle.$$  

Note that for any $z \in \mathbb{R}^{r+1}$, we have $f_j(z) = 0$ if and only if there exists some $b \in B_j$ such that $\langle b, z \rangle = 0$. Hence, $f_j(u_k) = 0$ if and only if there exists some $b \in B_j$ with $\langle b, u_k \rangle = 0$, i.e. if and only if $A_k \cap B_j \neq \emptyset$. Therefore, we have

$$f_j(u_j) \neq 0 \forall j \in [m]; \quad f_j(u_k) = 0 \forall k > j,$$  

as desired. This implies that the $f_j$’s are linearly independent.

Note that the $f_j$’s can be considered as (multivariate) polynomials in $\mathbb{R}[X_1, \ldots, X_{r+1}]$, of total degree exactly $s$. The vector space of these polynomials is spanned by
the set of monomials in $X_1, \ldots, X_{r+1}$ of total degree exactly $s$. The number of these is exactly
\[ \binom{r+s}{r}; \]
they are in 1-1 correspondence with the set of sequences of $s$ 'dots' and $r$ 'bars', where the number of dots between the $i$th and the $(i+1)$th bars corresponds to the power of $X_i$. Hence, $m \leq \binom{r+s}{r}$, as required.

We now sketch the idea behind the second proof. It turns out that we can construct an algebra $E$ over $\mathbb{R}$, equipped with a product $\wedge$, such that we can associate a vector $v_A \in E$ with any $r$-set $A \in [n]^{(r)}$, and a vector $v_B \in E$ with any $s$-set $B \in [n]^{(s)}$, where
\[ v_A \wedge v_B = 0 \iff A \cap B \neq \emptyset. \]
This immediately implies that the $v_A$'s are linearly independent in $E$. Indeed, suppose
\[ \sum_{j=1}^{m} c_j v_{A_j} = 0 \]
for some $c_1, \ldots, c_m \in \mathbb{R}$; we claim that all the $c_j$'s are zero. Suppose not, and let $l$ be minimal such that $c_l \neq 0$. Take the product of the left-hand side with $v_{B_l}$:
\[ \sum_{j=1}^{m} c_j (v_{A_j} \wedge v_{B_j}) = 0. \]
Since $A_k \cap B_l \neq \emptyset$ for all $k > l$, we have $v_{A_k} \wedge v_{B_l} = 0$ for all $k > l$, so this reduces to
\[ c_l (v_{A_l} \wedge v_{B_l}) = 0. \]
Since $A_l \cap B_l = \emptyset$, we have $v_{A_l} \wedge v_{B_l} \neq 0$, so $c_l = 0$, a contradiction. This proves the claim.

Finally,
\[ \{v_A : A \in [n]^{(r)}\} \]
will lie in a vector subspace of $E$ with dimension $\binom{r+s}{r}$. This will immediately imply Lovasz’ Theorem.

Now for the construction of our algebra. It will be the exterior algebra of the real vector space $\mathbb{R}^{r+s}$.

We may as well define the exterior algebra of an arbitrary vector space. Let $V$ be a vector space over a field $\mathbb{F}$, and let $T(V)$ denote the tensor algebra of $V$,
\[ T(V) := \mathbb{F} \oplus V \oplus (V \otimes V) \oplus (V \otimes V \otimes V) \oplus \ldots = \bigoplus_{k \geq 0} V \otimes^k; \]
equipped with the multiplication $\otimes$. Let $I$ denote the 2-sided ideal of $T(V)$ generated by the set $\{v \otimes v : v \in V\} \subset V \otimes V$. The exterior algebra $\Lambda(V)$ of $V$ is defined to be the quotient of the algebra $T(V)$ by $I$:
\[ \Lambda(V) := T(V)/I. \]
\[ ^2 \text{Recall that an algebra } E \text{ over a field } \mathbb{F} \text{ is an } \mathbb{F}-\text{vector space equipped with a multiplication operation, } \cdot, \text{ which is distributive over addition } ((x+y) \cdot z = x \cdot z + y \cdot z, z \cdot (x+y) = z \cdot x + z \cdot y \text{ for any } x, y, z \in E), \text{ and satisfies } \lambda(x \cdot y) = (\lambda x) \cdot y = x \cdot (\lambda y) \text{ for any } x, y \in E \text{ and } \lambda \in \mathbb{F}. \text{ Our algebras will all be associative: } x \cdot (y \cdot z) = (x \cdot y) \cdot z \text{ for any } x, y, z \in E. \]
Since $I \cap V^\otimes k$ is a subspace of $V^\otimes k$, we may define $\Lambda^k(V) = V^\otimes k / (I \cap V^\otimes k)$. Observe that $\Lambda(V)$ decomposes as a direct sum of these subspaces:

$$\Lambda(V) = \bigoplus_{k \geq 0} \Lambda^k(V).$$

The subspace $\Lambda^k(V)$ is called the $k$th exterior power of $V$; we have $\Lambda^0(V) = F$ and $\Lambda^1(V) = V$.

The multiplication on $\Lambda(V)$ (inherited from $T(V)$) is written $\wedge$:

$$(v + I) \wedge (w + I) := v \otimes w + I.$$ 

It is called the wedge-product, and turns $\Lambda(V)$ into an algebra over $F$; in particular, $\wedge$ is multilinear. It is associative, since $\otimes$ is.

By construction, $x \wedge x = 0$ for all $x \in \Lambda(V)$. This implies that $\wedge$ is anticommutative, meaning that $y \wedge x = -x \wedge y$ for all $x, y \in \Lambda(V)$. It follows that $\wedge$ is alternating, meaning that for any $x_1, \ldots, x_k \in \Lambda(V)$, and any permutation $\sigma \in S_k$, we have

$$x_{\sigma(1)} \wedge x_{\sigma(2)} \wedge \ldots \wedge x_{\sigma(k)} = \text{sign}(\sigma)(x_1 \wedge x_2 \wedge \ldots \wedge x_k).$$

Moreover, it follows that

$$x_1 \wedge x_2 \wedge \ldots \wedge x_k = 0$$

whenever the sequence $x_1, x_2, \ldots, x_k$ contains a repetition. This is the first property of $\Lambda(V)$ that we need.

We want to deal with the linear dependence of sequences (where we allow repetitions), rather than of sets, so it will be convenient to use the following (slightly fussy) definition

**Definition.** Let $V$ be a vector space over a field $F$. We say that a sequence $(x_1, \ldots, x_k)$ in $V$ is linearly dependent if it contains a repetition, or is linearly dependent as a set — in other words, if there exist scalars $\lambda_1, \lambda_2, \ldots, \lambda_k \in \mathbb{F}$, not all zero, such that

$$\sum_{i=1}^k \lambda_i x_i = 0.$$ 

Otherwise, it is said to be linearly independent.

It follows from the multilinearity and anticommutativity of $\wedge$ that

$$x_1 \wedge x_2 \wedge \ldots \wedge x_k = 0$$

whenever the sequence $(x_1, \ldots, x_k)$ is linearly dependent. Crucially, it turns out that this is an ‘if and only if’:

**Fact 1.** For any $x_1, \ldots, x_k \in V$, we have

$$x_1 \wedge x_2 \wedge \ldots \wedge x_k = 0$$

if and only if the sequence $(x_1, \ldots, x_k)$ is linearly dependent.
Remark. Thus, the wedge-product of vectors in \( \mathbb{F}^N \) can be considered as a ‘vector-valued version’ of the determinant function. Indeed, if \( x_1, \ldots, x_N \in \mathbb{F}^N \), then
\[
x_1 \wedge x_2 \wedge \ldots \wedge x_N = \det(x_1|x_2|\ldots|x_N)(e_1 \wedge e_2 \wedge \ldots \wedge e_N),
\]
where \( \{e_1, e_2, \ldots, e_N\} \) denotes the standard basis of unit vectors in \( \mathbb{F}^N \), and \( (x_1|x_2|\ldots|x_N) \) denotes the \( N \times N \) matrix with \( j \)th column \( x_j \).

Fact 1 follows from Lemma 5.

Let \( U \) be an \( N \)-dimensional vector-space over \( \mathbb{F} \), and let \( B = \{e_1, \ldots, e_N\} \) be a basis of \( U \). Then for each \( k \in \{0, 1, \ldots, N\} \), the set
\[
\{e_{i_1} \wedge e_{i_2} \wedge \ldots \wedge e_{i_k} : 1 \leq i_1 < i_2 < \ldots < i_k \leq N\}
\]
is a basis for \( \Lambda^k(U) \).

This is intuitively plausible; we give a proof in the appendix. In particular, we have
\[
\dim(\Lambda^k(U)) = \binom{N}{k}.
\]
This is the second (and final) property we need; our \( v_A \)'s will lie in the \( r \)th exterior power \( \Lambda^r(\mathbb{R}^{r+s}) \).

Remark. Since \( \Lambda^k(U) = 0 \) for all \( k > N \), we have
\[
\Lambda(U) = \bigoplus_{k=0}^N \Lambda^k(U),
\]
and therefore the set
\[
\{e_{i_1} \wedge e_{i_2} \wedge \ldots \wedge e_{i_k} : 0 \leq k \leq N, 1 \leq i_1 < i_2 < \ldots < i_k \leq N\}
\]
is a basis for \( \Lambda(U) \), which has dimension \( 2^N \). Hence, the natural basis of \( \Lambda^k(\mathbb{F}^N) \) is in 1-1 correspondence with \( [N]^{(k)} \), and the natural basis of \( \Lambda(\mathbb{F}^N) \) is in 1-1 correspondence with \( \mathcal{P}([N]) \).

We are now ready to give the second proof of Theorem 4. Our algebra \( E \) will be the exterior algebra of the real vector space \( \mathbb{R}^{r+s}, \Lambda(\mathbb{R}^{r+s}) \).

Second proof of Theorem 4. Take a set of \( n \) points \( Y = \{y_1, \ldots, y_n\} \subset \mathbb{R}^{r+s} \), in general position. For each subset \( S = \{i_1 < i_2 < \ldots < i_k\} \subset [n] \), define
\[
v_S = y_{i_1} \wedge y_{i_2} \wedge \ldots \wedge y_{i_k} \in \Lambda(\mathbb{R}^{r+s}).
\]
Then for each \( A \in [n]^{(r)} \), \( v_A \) lies in \( \Lambda^r(\mathbb{R}^{r+s}) \), which has dimension \( \binom{r+s}{r} \), and moreover, \( v_A \wedge v_B = 0 \) if and only if the sequence \( A \wedge B \) (\( A \) concatenated with \( B \)) is linearly dependent. Since the points of \( Y \) are in general position in \( \mathbb{R}^{r+s} \), this holds if and only if the sequence \( A \wedge B \) contains a repetition, i.e. if and only if \( A \cap B \neq \emptyset \). By the above discussion, Theorem 4 follows immediately.
3 Appendix

Proof of Lemma 5: We already know the ‘if’ part of Fact 1. It follows that if $U$ is a finite-dimensional vector space over $\mathbb{F}$, and $B = \{e_1, \ldots, e_N\}$ is a basis of $U$, then for each $k \in \{0, 1, \ldots, N\}$, the set

\[ \{e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_k} : 1 \leq i_1 < i_2 < \cdots < i_k \leq N\} \]

spans $\Lambda^k(U)$. It turns out that $\dim(\Lambda^k(U)) = \binom{N}{k}$, so the above set is a basis for $\Lambda^k(U)$. To prove rigourously that $\dim(\Lambda^k(U)) = \binom{N}{k}$, we argue as follows. Let $W$ denote the $\mathbb{F}$-vector space spanned by all finite formal $\mathbb{F}$-linear combinations of sequences $(e_{i_1}, e_{i_2}, \ldots, e_{i_k})$, where $1 \leq i_1 < i_2 < \cdots < i_k \leq N$. Define a map

\[ f : B^k \to W \]

by

\[ f(b_1, \ldots, b_k) = 0 \]

whenever $b_i = b_j$ for some $i \neq j$, and

\[ f(e_{i_{\sigma(1)}}, \ldots, e_{i_{\sigma(k)}}) = \text{sign}(\sigma)(e_{i_1}, \ldots, e_{i_k}) \]

whenever $1 \leq i_1 < i_2 < \cdots < i_k \leq N$ and $\sigma \in S_k$. Extend this to a multilinear map

\[ f' : U^k \to W; \]

then $f'$ is alternating, meaning that

\[ f'(u_{\sigma(1)}, \ldots, u_{\sigma(k)}) = \text{sign}(\sigma)f'(u_1, \ldots, u_k) \quad \forall u_i \in U, \sigma \in S_k. \]

Let

\[ \tilde{f} : U^\otimes k \to W \]

denote the induced linear map on $U^\otimes k$. It is easy to see that $I \cap U^\otimes k \subset \ker(\tilde{f})$, and therefore

\[ \dim(I \cap U^\otimes k) \leq \dim(\ker(\tilde{f})) = \dim(U^\otimes k) - \dim(\text{Im}(\tilde{f})) = N^k - \dim(W) = N^k - \binom{N}{k}. \]

Hence,

\[ \dim(\Lambda^k(U)) = \dim(U^\otimes k / (I \cap U^\otimes k)) = N^k - \dim(I \cap U^\otimes k) \geq \binom{N}{k}. \]

Therefore, $\dim(\Lambda^k(U)) = \binom{N}{k}$, as required. \qed